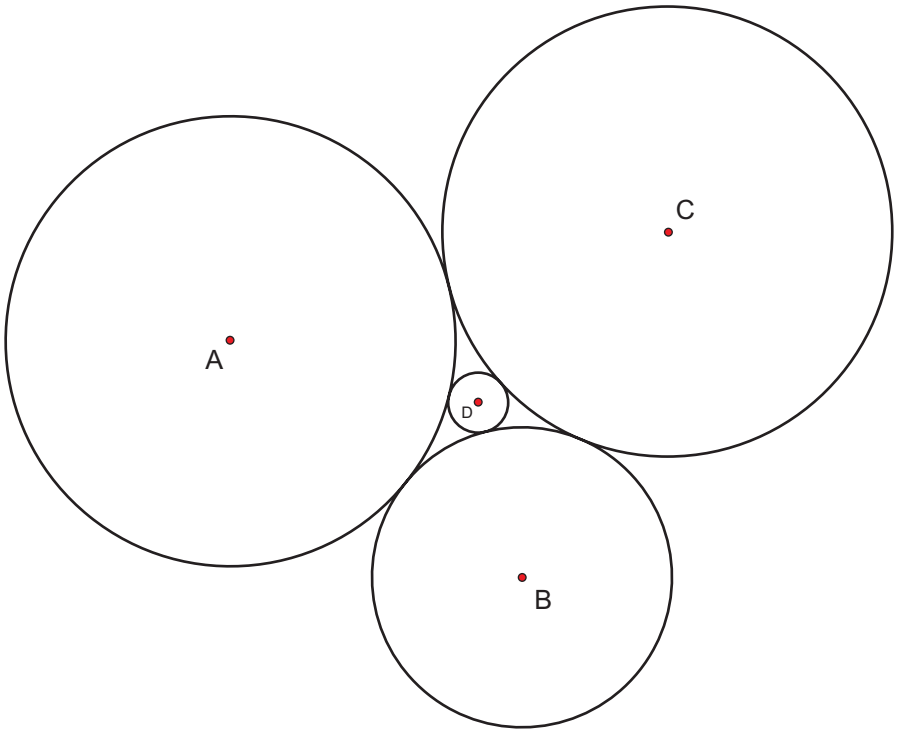
Circle Packing

NAME _____

In general, *curvature* is the amount by which a geometric object deviates from being *flat*. Mathematicians and geometricians study the curvature of all sorts of shapes—parabolas, exponential curves, even three-dimensional objects such as the Earth. As an introduction to curvature, this activity will allow you to explore the curvature of circles.

In the figure below, circles A, B, C, and D are mutually tangent to one another. Use this figure to answer Questions 1-4.



The curvature of a circle is equal to the **reciprocal of its radius**.

1. The radius of circle A is 15. The curvature of circle A is ____.

2. The radius of circle B is 10. The curvature of circle B is ____.

3. The curvature of circle C is 1/15. The radius of circle C is ____.

4. The curvature of circle D is 1/2. The radius of circle D is ____.

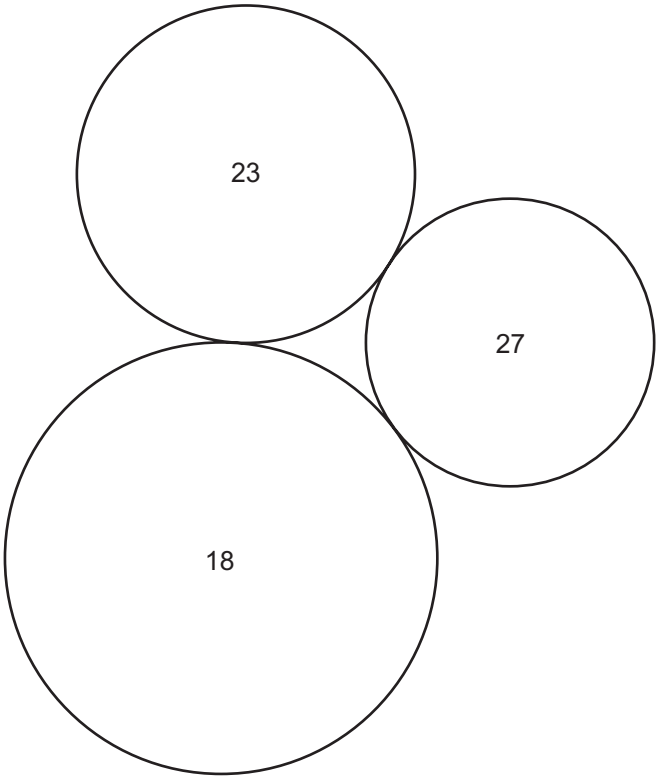
5. Which has greater curvature, a large circle or a small circle? Explain your answer.

In a letter to Princess Elizabeth of Bohemia, the French mathematician Rene Descartes described a solution to a problem involving the curvature of circles. **Descartes' Theorem** states that if four mutually tangent circles have curvature *a*, *b*, *c*, and *d*, then…

$$(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2)$$

6. Show that Descartes' Theorem holds for the four circles A, B, C, and D described in Questions 1-4.

The three circles shown to the right have curvatures of 18, 23, and 27, as indicated.



7. Two different circles could be drawn that are tangent to the three circles shown to the right.

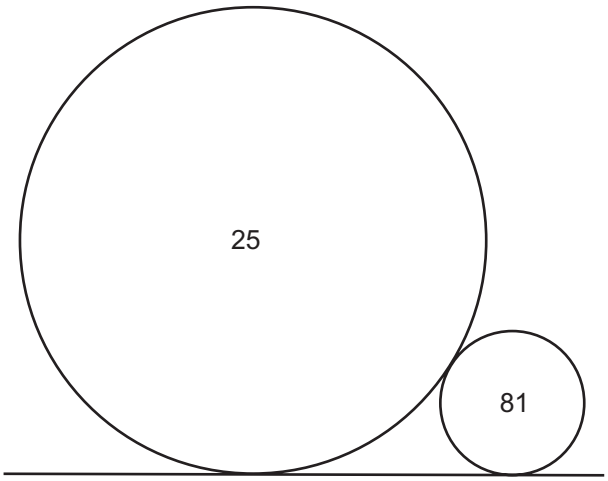
   a. Draw a circle that is tangent to each of the three circles on the inside. This circle is said to be *internally* tangent.

   b. Draw a circle that is tangent to each of the three circles on the outside. This circle is said to be *externally* tangent.

8. Use Descartes' Theorem to find the curvature of a fourth circle that would be tangent to these three circles.

9. You should have gotten two different answers for the curvature to Question 8. (If you didn't, go back and check your work.) Each curvature value corresponds to one of the circles you drew in Question 7. Which circle has which curvature? Explain how you know.

10. There are two different circles that could be drawn tangent to each of the two circles and to the line shown below. Draw them.

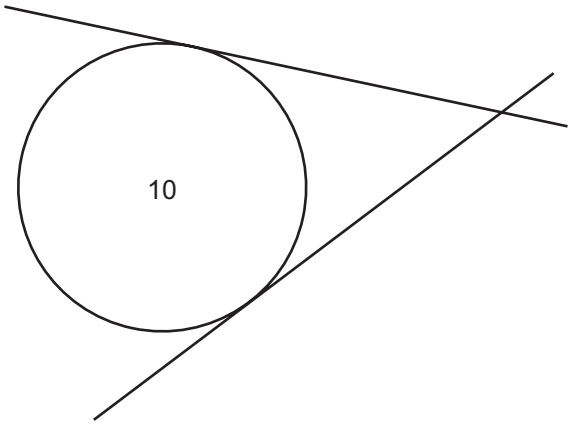


11. What is the curvature of a straight line? (*Hint:* Think about continually increasing the size of a circle. As the circle gets larger and larger, a small portion of the circumference will look like a straight line. What is the radius of this infinitely large circle? What is the reciprocal of the radius?)

12. As indicated above, the two circles have curvature 25 and 81, and you found the curvature of a straight line in Question 11. Using these three values and Descartes' Theorem, find the curvature of the two circles that you drew in Question 10.

13. Two circles, each with curvature 20, are tangent to a straight line. Use Descartes' Theorem to find the curvature of a fourth circle that is tangent to both circles and to the line. Interpret the results of your calculation.

As you saw, Descartes' Theorem can be used to find a tangent circle when two circles and a straight line are given. However, the theorem **cannot be used** to find a tangent circle when only one circle and two straight lines are given, as shown below.



14. Use the equation from Descartes' Theorem to find the radius of a circle tangent to the circle and two lines shown above. That is, let $a = 0$, $b = 0$, and $c = 10$. What happens? Interpret the results.

15. How many different circles could be drawn that are tangent to the two lines and circle shown? Draw at least one of them on the figure above.