This brainteaser was written by Derrick Niederman.

Find four positive integers a, b, c, and d such that the product $abcd$ is equal to the sum of the squares, $a^2 + b^2 + c^2 + d^2$.

$$\boldsymbol{abcd = a^2 + b^2 + c^2 + d^2}$$

What? That's too easy, you say? You're probably right. But can you find four different solutions —

- One that uses the same number four times?
- One that uses the same number three times?
- One that uses the same number twice?
- And, one that uses four different numbers?

Solution:

Same number four times, $2 \times 2 \times 2 \times 2 = 2^2 + 2^2 + 2^2 + 2^2 = 16$.

Same number three times, $6 \times 2 \times 2 \times 2 = 6^2 + 2^2 + 2^2 + 2^2 = 48$.

Same number two times, $22 \times 6 \times 2 \times 2 = 22^2 + 6^2 + 2^2 + 2^2 = 528$.

Four different numbers, $262 \times 22 \times 6 \times 2 = 262^2 + 22^2 + 6^2 + 2^2 = 69{,}168$.

The solution (2, 2, 2, 2) is rather easy to find. It can be found using a guess-and-check strategy, or you can solve it algebraically with the equation $x^4 = 4x^2$. That equation simplifies to $x^2 = 4$, so $x = 2$. (It turns out that $x = -2$ is also a solution to that equation, but the problem asks for positive integers.)

In general, if (a, b, c, d) is a solution to the equation, then ($bcd – a$, b, c, d) is also a solution. That is, the value of a can be replaced by the value of $bcd – a$. It may not be obvious why this is true, but the following algebra shows that if a is replaced by $bcd – a$, then the equation still holds true:

$$abcd = a^2 + b^2 + c^2 + d^2$$

$$(bcd - a)bcd = (bcd - a)^2 + b^2 + c^2 + d^2$$

$$bcd^2 - abcd = bcd^2 - 2abcd + a^2 + b^2 + c^2 + d^2$$

$$bcd^2 - abcd - \left(bcd^2 - 2abcd\right) = bcd^2 - 2abcd - \left(bcd^2 - 2abcd\right) + a^2 + b^2 + c^2 + d^2$$

$$abcd = a^2 + b^2 + c^2 + d^2$$

Consequently, if (a, b, c, d) = (2, 2, 2, 2), then $bcd – a = 2 \times 2 \times 2 – 2 = 6$ can replace a to yield the solution (6, 2, 2, 2). This solution uses one number three times.

The order of the numbers is unimportant, so (6, 2, 2, 2) = (2, 6, 2, 2) = (2, 2, 6, 2) = (2, 2, 2, 6). Taking the second of these as (a, b, c, d) = (2, 6, 2, 2), then $bcd – a = 6 \times 2 \times 2 – 2 = 22$ can replace a to yield the solution (22, 6, 2, 2). This solution uses one number twice.

By similar reasoning, $22 \times 6 \times 2 – 2 = 262$ can replace a in (2, 22, 6, 2) to give the solution (262, 22, 6, 2), which uses four different numbers.