This brainteaser was written by Derrick Niederman.

The Fibonacci sequence is shown below, with each term equal to the sum of the previous two terms. If you take the ratios of successive terms, you get 1, 2, 3/2, 5/3, 8/5, 13/8, and so on. But as you proceed through the sequence, these ratios get closer and closer to a fixed number, known as the Golden Ratio.

1, 1, 2, 3, 5, 8, 13, ...

Using the rule that defines the Fibonacci sequence, can you determine the value of the Golden Ratio?

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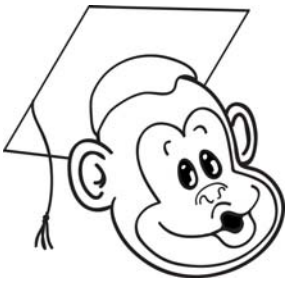

Solution: $\mathbf{\frac{1+\sqrt{5}}{2}}$**.**

The concept of approaching a limit is usually introduced in calculus, but we can investigate it here. Denote the nth Fibonacci number as F_n. If the sequence of Fibonacci ratios converges to a fixed limit a, as suggested, then for large values of n, $F_{n+1} \approx a \cdot F_n$ and $F_{n+2} \approx a^2 \cdot F_n$.

Since each term is equal to the sum of the previous two, this gives $F_{n+2} = F_{n+1} + F_n$ which implies $a^2 \cdot F_n = a \cdot F_n + F_n$, which simplifies to $a^2 - a - 1 = 0$. Applying the quadratic formula to this equation yields:

$$a = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$$

Only the positive result is reasonable in this situation, so the Golden Ratio is approximately 1.618.

Of course, the equations used above are only approximations, and we have assumed that a limit exists in the first place. However, if a limit does indeed exist, then it must equal the given value.

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