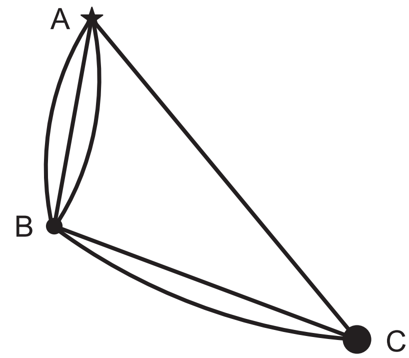
This brainteaser was written by Derrick Niederman.

There are 59 different routes from Arlington to Bedford, including those that go through Cambridge. There are 39 different routes from Bedford to Cambridge, including those that go through Arlington.

How many routes are there from Arlington to Cambridge?

Solution: 51.

Before jumping in to this problem, consider a simpler example. Assume that there are three direct routes between A and B, two direct routes between B and C, and one direct route between A and C, as shown to the right.



Then, there are:

- $3 + 2 \times 1 = 5$ routes between A and B
- $2 + 3 \times 1 = 5$ routes between B and C
- $1 + 2 \times 3 = 7$ routes between A and C

That is, the number of routes between two cities is equal to the number of direct routes between those cities, plus the product of direct routes between the other cities.

So, for this problem, let x equal the number of direct routes between Arlington and Bedford, y the number of direct routes between Bedford and Cambridge, and z the number of direct routes between Arlington and Cambridge. This gives the following equations:

$$A \rightarrow B\text{: } 59 = x + yz$$
$$B \rightarrow C\text{: } 39 = y + xz$$

Adding and subtracting these equations gives

$$98 = (x + y)(z + 1)$$
$$20 = (y - x)(z - 1)$$

The proper factors of 98 are 1, 2, 7, 14. The proper factors of 20 are 1, 2, 5, 10. The above equations imply that $z + 1$ divides evenly into 98 and $z - 1$ divides evenly into 20; because these two numbers differ by two, it must be that $z + 1 = 7$ and $z - 1 = 5$, so $z = 6$.

Substituting $z = 6$ into the above equations, we get

$$59 = x + 6y$$
$$39 = y + 6x,$$

which yields $x = 5$ and $y = 9$. Therefore, the total number of routes between Arlington and Cambridge, given by $z + xy$, is $6 + 5 \times 9 = 51$.