

Arithme-Tic-Toc Answer Key

Complete the Mod-5 Clock Tables below for addition and multiplication. Pay special attention, looking for any patterns or relationships that exist. Below the tables, list the observations that you make, and be prepared to support your observations and discuss them with other classmates.

ADDITION TABLE					
+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

MULTIPLICATION TABLE					
×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Observations:

Numbers repeat along diagonals.

Each number is used in every row.

Each number is used in every column.

The numbers in each row and column proceed in order, then repeat.

Observations:

There is a row of 0's.

There is a column of 0's.

Each number is used in every row.

Each number is used in every column.

Use a Mod-12 Addition Table to answer Questions 1-3.

1. Examine the diagonals in the table. Describe any patterns that you observe.

There are many observations to be made.

- *Each diagonal (with a positive slope) of the table consists of the same value.*
- *Only even values are found along the main diagonal.*
- *The diagonals that run parallel to the main diagonal have entries that are all even or all odd.*
- *The main diagonal is a line of symmetry.*
- *Diagonals opposite each other running perpendicular to the main diagonal, when added to each other, have a sum of 10 (except the middle diagonal of all 11's).*

2. What is an additive inverse? Give an example from regular arithmetic. Then give two examples from the table.

An additive inverse is a number that, when added to another number, results in 0. For instance, the additive inverse of 7 is -7, because $7 + (-7) = 0$. From the mod-12 table, 2 is the additive inverse of 10, because $2 + 10 = 0 \pmod{12}$.

3. Below are definitions for each of the properties of real numbers. Give an example from the addition table for each property.

Identity: $a + 0 = a$ Commutative: $a + b = b + a$ Associative: $a + (b + c) = (a + b) + c$

The additive identity is found in either the first row or first column. All operations in the table are commutative and associative.

Use a Mod-12 Multiplication Table to answer Questions 4-9.

4. In regular arithmetic, what is the multiplicative inverse of 7? How do you determine if two numbers are multiplicative inverses? Extend this idea to the table. Identify all inverses. What is different about these inverses in the tables compared to the multiplicative inverse in regular arithmetic?

The inverse of 7 is $1/7$. Two numbers are multiplicative inverses if their product is 1. The inverses on the mod 12 table are 1, 5, 7, and 11. Each of these numbers is its own inverse.

5. Notice that some rows and columns contain each value 1 through 11 exactly once. Explain where and why this happens.

This happens in Rows 1, 5, 7, and 11, because 1, 5, 7, and 11 are each relatively prime to 12.

6. Describe any symmetry.

The main diagonal is a line of symmetry for the entire table, excluding the row and column of zeros. The numbers along the main diagonal and the diagonal perpendicular to it, again excluding the row and column of zeros, exhibit symmetry. These patterns are palindromic sequences of numbers.

7. Verify that the table is commutative for all values.

Symmetry about the main diagonal provides the proof for the commutative property. The table is also associative, but proving this is more difficult. This is saved as an extension activity.

8. In regular arithmetic, $6 \times 17 = 102$. Determine a process, using the distributive property of addition over multiplication, to show that $6 \times 17 = 6 \pmod{12}$.

Because you are working mod 12, separate any number greater than 12 into a sum involving multiples of 12. In this case, use $17 = 12 + 5$, as follows:

$$\begin{aligned}6 \times 17 &= 6 \times (12 + 5) \\ &= 6 \times (0 + 5) \\ &= (6 \times 0) + (6 \times 5) \\ &= 0 + 6 \\ &= 6 \pmod{12}\end{aligned}$$

Other sums of 17 could be used instead (such as $11 + 6$ or $9 + 8$), but in order to use the table, the largest addend should be a multiple of 12. Computations with multiples of 12 are easier, because all multiples of 12 are congruent to 0 mod 12.

9. Notice that the numbers in some rows are in the exact reverse order of the numbers in another row. Name the row pairs for which this happens, and explain why it happens.

The row pairs are 1-11, 2-10, 3-9, 4-8, and 5-7. Numbers occur in reverse order for Row n and Row $(12 - n)$. This happens because $12 - n = -n \pmod{12}$.

10. Many of the things we use each day involve modular arithmetic, such as the hours on a clock. Describe at least three other situations that use modular arithmetic.

There are many examples:

- *Multilevel light switches: when you turn the switch once, the light turns on; turn the switch again, it becomes brighter; turn it a third time, the light turns off; then, the cycle repeats.*
- *Days of the week repeat on a 7-day cycle.*
- *Changing the input mode on your television: Video 1, Video 2, DVD, Game, Tuner. Then, back to Video 1.*
- *Imaginary unit: $i, i^2 = -1, i^3, i^4 = 1, i^5 = i, i^6 = -1, \dots$*
- *Playing songs on a CD. A CD with 14 songs set to “repeat” mode means that the 20th song you hear will be the 6th song played a second time.*
- *Sine, cosine, and tangent periods.*
- *Reference angles: $450^\circ = 90^\circ \text{ mod } 360^\circ$.*
- *Stop lights: red, yellow, green, red, yellow, green,*
- *Counting off by 3, 4, or 7 to form groups.*
- *Odometers “roll over” after they hit a certain mileage and return to 000000.*
- *The four seasons: Winter, Spring, Summer, Fall.*